Math 113-008, Exam 1
21-24 September 2007
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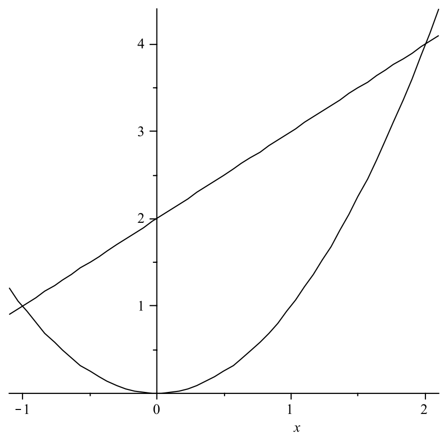
Name ____________________
Row____________________

1. (20%)

   (a) Carefully sketch the region between the curves $y = x^2$ and $y = x + 2$. The rest of the problem depends on this sketch.



   (b) Set up an integral for the area of region bounded by the curves in part (a). DO NOT EVALUATE THE INTEGRAL!

$$\int_{-1}^{2} (x + 2 - x^2)\ dx$$

   (c) Set up an integral for the volume when the region is rotated about the x−axis. DO NOT EVALUATE THE INTEGRAL!

$$\pi \int_{-1}^{2} ((x+2)^2 - x^4)\ dx$$

   (d) Set up an integral for the volume when the region is rotated about the line $x = 2$. DO NOT EVALUATE THE INTEGRAL!

$$2\pi \int_{-1}^{2} (2 - x)(x + 2 - x^2)\ dx$$

2. (10%) Show that the volume V of a cone with radius r and height h is given by $V = \frac{1}{3}\pi r^2 h$.

$$\pi \int_0^h \left(\frac{r}{h}x\right)^2\ dx = \frac{\pi r^2}{h^2} \int_0^h x^2\ dx = \frac{\pi r^2}{h^2}\frac{h^3}{3} = \frac{1}{3}\pi r^2 h$$

3. (10%) Suppose that 16 foot-lbs of work is needed to stretch a spring from its natural length of 12 inches to 18 inches. How much work is required to stretch the spring from 18 inches to 24 inches?

The spring is stretched 1/2 ft from rest position. $\int_0^{1/2} kx\, dx = \frac{k}{8} = 16$ So $k = 128$.

From 18 inches to 24 inches means 1/2 ft to 1 ft beyond rest position. $\int_{1/2}^{1} 128x\, dx = \frac{128}{2}\left(1 - \frac{1}{4}\right) =$ $64\left(\frac{3}{4}\right) = 48$ ft-lbs.

4. (10%) A tank has the shape of a surface generated by revolving the parabolic segment $y = x^2$ for $0 \le x \le 3$ about the y-axis (measurement in feet). If the tank is full of a fluid weighing 100 pounds per cubic foot, set up an integral for the work required to pump the contents of the tank to a level 5 feet above the top of the tank. DO NOT EVALUATE THE INTEGRAL!

$$100\pi \int_0^9 (14 - y)(\sqrt{y})^2\, dy \quad \mathrm{ft} - \mathrm{lbs}$$

5. (10%) Find the average value of the function $f(x) = x^2$ for $1 \le x \le 3$.

$$\frac{1}{3-1}\int_1^3 x^2\, dx = \frac{1}{2}\frac{x^3}{3}\Big|_1^3 = \frac{1}{6}(27 - 1) = \frac{13}{3}$$

6. (40%) Evaluate the following integrals:

(a) $\int xe^{-x}\,dx$

$$u = x \qquad dv = e^{-x}dx$$

$$du = dx \qquad v = -e^{-x}$$

$$\int xe^{-x}\,dx = -xe^{-x} - \int -e^{-x}\,dx = -xe^{-x} - e^{-x} + C$$

(b) $\int e^{2\theta}\sin 3\theta d\theta$

Use integration by parts twice.

$$\frac{e^{2\theta}}{13}(2\sin(3\theta) - 3\cos(3\theta)) + C$$

(c) $\int \sin^5 x \cos^2 x\,dx$

$$\int \sin^5 x \cos^2 x\,dx = \int \sin^4 x \cos^2 x(\sin x\,dx) = \int (1{-}\cos^2 x)^2 \cos^2 x(\sin x\,dx) = \int (1{-}\cos^2 x)^2 \cos^2 x(\sin x\,dx)$$

$$= -\int (\cos^2 x - 2\cos^4 x + \cos^6 x)(-\sin x\,dx)\,dx = -\left(\frac{\cos^3 x}{3} - 2\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7}\right) + C$$

$$= -\frac{\cos^3 x}{3} + 2\frac{\cos^5 x}{5} - \frac{\cos^7 x}{7} + C$$

(d) $\int_0^{\pi} \cos^4 2x\,dx$

$$\int_0^{\pi} \cos^4 2x\,dx = \int_0^{\pi} \left(\frac{1+\cos(4x)}{2}\right)^2 dx = \frac{1}{4}\int_0^{\pi} \left(1 + 2\cos(4x) + \cos^2(4x)\right)\,dx$$

$$= \frac{1}{4}\int_0^{\pi} \left(1 + 2\cos(4x) + \cos^2(4x)\right)\,dx = \frac{1}{4}\int_0^{\pi} \left(1 + 2\cos(4x) + \frac{1+\cos(8x)}{2}\right)\,dx$$

$$\frac{1}{4}\int_0^{\pi} \left(\frac{3}{2} + 2\cos(4x) + \frac{\cos(8x)}{2}\right)\,dx = \frac{3\pi}{8}$$